Tessellations by Polygons in Mathematics Education

Key words: tessellation, polygonal shapes, interior angles in polygons, quadrilaterals, planar imagery;

Abstract: This paper presents assumptions and partial results of a pilot study concerning a possible implementation of polygonal tessellations in mathematics education. The first part of the contribution consists of historical facts, background of tessellation, aims and methods of research. This is followed by illustrations from the interviews in which selected real life's problems were solved and pictures of polygonal tessellations created by pupils.

INTRODUCTION

A tessellation of a plane is a family of sets - called tiles (or cells) - that cover the plane without gaps or overlaps (Grübaum, Shephard 1977). Tessellations are also called tilings, pavings, parquetings or mosaics (the name 'tiling' is usually used if the tiles are translation equivalent to certain prototile).

The word tessellation is derived from English verb tessellate which comes from Greek word τέσσαρες (tessares) meaning 'four'. The Greeks called the die τέσσαρα because any side of it has four edges. Latin word for die is tessera and its diminutive is tesserula. In the dictionary (Webster's New Collegiate Dictionary, 1989) the word 'tessera' is used for a small tablet of wood, ivory, etc. used as a token, ticket, tabel in ancient Rome.

The art of creating tessellation has been known since the origin of the history of civilization. When people began to build their houses and fortifications, they tried to fill space or plane. The first houses, churches and castles were built by broken stones which made up random tessellations. Later, arrangements of prismatic stones and bricks possessed a large regularity. In Gothic architecture, window-panes (mosaics with colored glass pieces connected by lead strips) have occurred.

Different sorts of parquetings, complicated covering of ceilings or room's walls by tiles, or pavings of paths and places are samples of tessellations documenting human creativity.

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Every known human society made use of tilings and patterns in some form or another. People's portraits and nature's scenes often occurred on intricate mosaics from Mediterranean region. On the other hand, Islamic religion strictly prohibits such depictions of humans and animals which might result in idol-worship, so that Moors and Arabs created art utilizing a number of primary forms: geometric, arabesque, floral and calligraphic. Moorish architecture in Spain and Islamic culture in the Middle East have shown grateful samples of planar tessellations with rich ornaments. The palace Alhambra in Spanish Granada is one of the best-known and most-preserved relic of the Islamic civilization on Pyrenees peninsula.

Planar tessellations occur also in the modern art of the twentieth century, namely in creations of Dutch artists M. C. Escher (1898-1972) and P. Mondrian (1872-1944). In fact, Escher was fascinated and deeply inspired by Alhambra's mosaics and later he used some motifs derived from them in his prints. He used mathematical imagination as tool for his graphics, he often filled plane by figures of people and animals (Day and Night, 1938) or connected filling of plane and space (The Reptiles, 1943). He said about his work (Ranucci and Teeters, 1977): "Although I am absolutely without training or knowledge in the exact sciences, I often seem to have more in common with mathematicians than with my fellow artists.” Mondrian often used nets and grids to depict reality. In this way he tried to avoid differences between figure and background, between mass and antimass (Composition with net 8, Chessboard with dark colors, 1919; Broadway Boogie Woogie, 1942-1943) (Golding, 2000).

The first deep mathematical study of tessellations, Harmonices Mundi, was written in 1619 by Johannes Kepler. He described geometric properties of planar tessellations by regular polygons. However, his contribution to astronomy was so monumental that his geometric investigations were largely forgotten for almost 300 years.

In 1975, papers about planar tiling by pentagons were published in the Scientific American. It has been an interesting problem for mathematicians because there has not been any rule for their general construction. Marjorie Rice, a San Diego housewife with no formal education in mathematics but with enormous enthusiasm, has started to find out new and until unknown types of pentagons tiling the plane.

Recently B. Grünbaum and G. C. Shephard elaborated the detailed theory of planar tessellations and symmetries of the patterns (e.g. Grünbaum and Shephard, 1987).

**MATHEMATICAL BACKGROUND**

Some part of the mathematical theory concerning tessellations is elementary, but it contains a rich supply of interesting and surprising problems which can be solved at various age levels of pupils. In this area, we focus on the problem of some types of the planar tessellations created by polygons.

Let us consider three following mathematical statements important for this research (the first one is a part of mathematics curriculum and is given without proof here):

**Statement 1:** The sum of all interior angles in every quadrilateral is equal to 360°.

**Statement 2:** Every quadrilateral tessellates the plane by itself.
The existence of the tessellation built up of the arbitrary identical quadrilateral tiles is a content of this statement.

Proof:
Let \( \alpha, \beta, \gamma, \delta \) are the interior angles of a general quadrilateral. Their sum is equal to \( 360^\circ \). Every quadrilateral can be arranged in the tessellation as shown in the picture:

Four quadrilaterals meet at each vertex and the corresponding interior angles form together the angle of \( 360^\circ \), so that there are no gaps and overlaps. (The same statement holds for the triangles; only some specific types of other polygons tessellate the plane by themselves.)

Statement 3: *Only three regular polygons tessellate the plane by themselves: triangles, squares and hexagons.*

Proof (for the equilateral triangles):
All interior angle of equilateral triangle is equal to \( 60^\circ \). Six triangles meet at each vertex and six corresponding interior angles form the angle of \( 360^\circ \) (= \( 6 \times 60^\circ \)). So that there are no gaps and overlaps.

PROPOSED RESEARCH QUESTIONS

I worked twice with my pupils (14/15 years, 15/16 years) on problems concerning tessellations (in 2002 and 2003). Their groups were large in both cases and, moreover, several other problems have to be covered in the lessons, so that there was not sufficient time for a more detailed examination. However, this experience could serve as a basis for the present qualitative experiment.

We focus in the research on detecting some cognitive and interactive phenomena which can be divided into five groups. They can be characterised as the answers to the following questions:

**Cognitive phenomena**

1. *Means of expressions and bilingualism*

What verbal and nonverbal communication tools do children use?
How successful is the communication in pairs and between pairs and experimenter?
The interviews were held in Czech (solvers) and Slovak (experimenter); how do children manage to overcome the bilingualism? (These languages are closed enough, but there are some differences, e.g. in the terminology of quadrilaterals: *square* - čtverec/štvorec, *rectangle* - obdélník/obdĺžnik, *general* - obecný/všeobecný,...)

2. *Planar imagery*

There is much information about the imagery and the spatial imagery available, but the
concept of 'planar imagery' is not much known and used. In our research we try to find out the things characterize this phenomenon.
The planar imagery could be defined as “... the basic psychic function, which is important for the psychic visualization of the planar events that are not actual, in the constructive and reconstructive meanings ...”.

(according Půlpán et al., 1992)

Which tools do children choose to solve the problems?
What strategy do they use - what do they do? Do they improve their strategy by preceding experience? Do they improve their strategy if they are unsuccessful?
How many different arrangements of tiles do they manage to find?

3. Mathematical content
Which mathematical concepts do children use? (the terminology of the quadrilaterals, the properties of figures and angles, ...) Do they understand them? Do they find out why every quadrilateral can be used as tile?

4. Proofs and justifying
Do children justify their answers? Do they feel the need of a proof? How do they prove their statements?

Interactive phenomena
Interpersonal relations
How do children work together? What are their mutual relations as partners in looking for the solution of the problem? How do they react when they are not successful and are told that the problem does have a solution?

The points 2 and 4 seem to be the most important in this research.

METHODOLOGY
The quadrilaterals have been chosen as tiles because some of such tessellations are the most frequently occurring in the real life (square tiles, rectangular parquets) and the mathematical solution concerning them can be easy to find for the majority of children. Five used types of quadrilaterals are those which are the most frequently handled in the school geometry, only the sixth one (a general nonconvex quadrilateral) is uncommon. (Operations with triangles and regular types of other polygons are made-up on similar principles.)

In problems, manipulating with models of quadrilaterals proceeds from the simplest shape - square - to general convex and nonconvex shapes. Thus pupils can repeat hierarchy of the quadrilaterals or it can be the opportunity to begin with it (during solving problems children also observed the properties of the quadrilateral figures - parallelism of the opposite sides, length of the sides, interior angles, ...).
Semi-structured interviews with the pairs of pupils from the age of 13 to 16 years were carried out separately. Interviews took time from 17 to 27 minutes and took place in classroom after the lessons. If the children were communicative they were not interrupted, otherwise supplementary questions were posed (why did you do so? can you explain it? ...).

The tools for solving the problems were set up on the table and none of them were preferred (the children were free to choose convenient tools for their work). The tools can be divided into two groups: the main tools (square, colored and white papers of the format A4, paper models of quadrilaterals, pencils) and the supplementary tools (scissors, ruler, pens); “the building set” is my working name for the paper models of quadrilaterals. It consisted of 10 - 14 figures for every chosen types of the quadrilaterals (square, rectangle, parallelogram, trapezium, general convex and nonconvex quadrilateral) and was made of the cardboard by the experimenter before the interviews.

For the subsequent analysis, the interviews have to be recorded; video camera and dictaphone are unavoidable.

![Quadrilaterals contained in 'the building set'](image)

Required problems are formulated as the questions concerning the real life that can be answered by YES-NO and substantiated e. g. by a picture or a mosaic created by the paper quadrilaterals.

Problems:
1. *When tiling the wall in the bathroom, can we use identical tiles of a square shape?*
2. *Can we use identical parquets of a rectangular shape to cover the floor?*
3. *Can we use identical parquets of a parallelogram shape?*
4. *Can we use identical parquets of a trapezoid shape?*
5. *Can we use identical parquets of such a shape?* (a general convex quadrilateral)
6. *Can we use identical parquets of such a shape?* (a general nonconvex quadrilateral)

(The concepts 'convex' and 'nonconvex' are unknown for pupils of such age levels, so these quadrilaterals are pointed them.)

*The first four problems have several solutions, hence the children were asked: Is the proposed tiling the unique possible one? Can these tiles (parquets) be arranged otherwise?*

After solving these particular problems, the pupils were asked the following questions:

- Is it possible to use identical parquets of an arbitrary quadrilateral shape?
  - If YES, why?
  - (Did you examine with all possible quadrilateral shapes?)
  - If NO, find out parquets of such a quadrilateral shape which we cannot use to cover the floor.
The first question concerns the square tiles in the bathroom, the other ones concern the parquets; why are the problems formulated that way?

After closing the interviews, the pupils were asked to fill a questionnaire and to do a homework. Questions concern their mathematical efforts, dealings with their cooperative partner and relation to the subject of mathematics. The answers (especially questions 3. and 4., see below) are subsequently analysed. The homework concerns tessellations by other polygons (triangle, pentagon, hexagon, ...). The observation during the interview could be slightly subjective, hence it is not regarded as the most important (but it can help to explain some phenomena, e.g. observed nervousness of children could explain their approach to the problem). The basic information about pupils was acquired from the teacher. The children's answers compared mutually and with our assumed (correct) answers.

Questionnaire
1. What is the mathematics mark on your school report?
   Are you satisfied with your results in mathematics or do you want to improve?
2. How do you get along with a boy/girl that you solved problems with?
   Are you good friends?
3. Did you like the solved problems? Are they different from those ones solved in mathematics lessons? What is the difference?
4. Which quadrilateral figure does seem to be the most interesting for you and why?

Homework
1. Have you got the tiled bathroom or kitchen at home? Have you got the parquets at home? If yes, draw them.
2. Think about parquets of another shapes, e.g. triangular, pentagonal, hexagonal,...and draw such parquetings.

The resulting drawings of the tessellations by the polygons different from the quadrilaterals will be analysed by the method of the case study.

(In my previous experiences pupils created designs of the tessellations for wall, path or floor tilings. They used mostly tessellations by very complicated shapes which could not be simply justified mathematically.)

PARTIAL RESULTS

Five pairs of children were interviewed. Their names, ages and simple characteristics of pairs are in the following table. Children cooperated very well in pairs; I suppose that teachers chose friends for pairs (it was not my decision).

<table>
<thead>
<tr>
<th>Names</th>
<th>Age and sex</th>
<th>Date of experiment</th>
<th>Comments (experimenter's observation)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Michael (M)</td>
<td>13, M</td>
<td>February 2004</td>
<td>easy to work with, very communicative pair</td>
</tr>
<tr>
<td>Petr (P1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Kristína (K)</td>
<td>13, F</td>
<td>February 2004</td>
<td>easy to work with, but they were a little bit bored by this activity and they laughed often</td>
</tr>
<tr>
<td>Tereza (T)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Names</td>
<td>Age and sex</td>
<td>Date of experiment</td>
<td>Comments (experimenter's observation)</td>
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<tr>
<td>------------------</td>
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<td>---------------------------------------------------------------------------</td>
</tr>
<tr>
<td>Lukáš (L) Jan (J)</td>
<td>16,15, M</td>
<td>May 2004</td>
<td>they were very nervous (I suppose, because of not knowing me )</td>
</tr>
<tr>
<td>Hana (H) Veronika (V)</td>
<td>15, F</td>
<td>May 2004</td>
<td>even when H was the main verbal communicator of pair, V was not the passive solver only</td>
</tr>
<tr>
<td>Gábina (G) Petr (P2)</td>
<td>14, F, M</td>
<td>May 2004</td>
<td>easy to work with, the both children reacted quickly to my questions</td>
</tr>
</tbody>
</table>

**Illustrations from the interviews**

From the transcriptions of the interviews, the following extracts were chosen to illustrate the children’s thinking and solving process when answering two selected questions.

The question *Is it possible to use identical parquets of an arbitrary quadrilateral shape?* was answered as follows:

**M:** Yes, it could be, but it goes from difficult to even more difficult.

**P1:** Sure, it would be possible. But it would be very complicated.

... 

**P1:** *So that if you have these angles, so you can put them together.*

(This sentence seems to be the closest to the formal solution.)

**J:** ...I don't know...(long pause)... if it were perhaps some regular one... perhaps parallelograms fitting together... but with some arbitrary quadrilateral, then it wouldn't go... it wouldn't fit together.

**H:** Yes, I think so. Except this one (she points to the unsuccessful tessellation by nonconvex quadrilateral)... so that... no... (they laugh).

**V:** Just those ones which we managed.

**G:** No.

The question *The first question concerns the square tiles in the bathroom, the other problems concern the parquets; suppose, why do I formulate problems that way?* was answered as follows:

**K:** Because... In fact... Because this one (the tessellation by nonconvex quadrilateral) wouldn't go for the wall rather like parquets.

**T:** Rather it would be all bad useable.

**K:** But it would be strange to look at.

**T:** ... Maybe it would be bad to handle.

**K:** It's also complicated and these (the tessellation by squares) are so simple.

**L:** That is... that walls are usually tiled by square tiles. That it isn't so difficult to arrange, to find that combination and the square has all sides of the same length.

**H:** There are usually these tiles in the bathroom... the square ones. It would look
very strange if there were these ones (she points to the nonconvex quadrilateral's tessellation).

V: It's the simplest way, isn't it?

G: Because these tiles are mainly the squares or of the rectangular shape...

P2: I haven't seen such tiles as yet (he points to the nonconvex quadrilateral tessellation).

G: Neither me.

... 

G: They seems to me (she thinks about) ...

P2: ...so kitscher.

Notes on the course of the interviews

My interest in mathematics was known to children so that they were rather surprised by my first posed question. (Perhaps they expected “something more mathematical”.) Also further problems sometimes evoked pupils' laughing. But except the pair T+K, the other children wrote or told that they were interested in the required problems.

All pairs were communicative. Even when some of them were little nervous at the beginning of the interview, there were no problems to work with them. Sometimes children communicated between themselves in a low voice or by the “eye contacts”, they also used hands as a complementary communicative tool. There was no problem with the bilingualism.

The answers to the required problems can be substantiated by means of using 'the building set' (the paper models) or drawing the corresponding tilings. Only one pair (L+J) solved the first three problems by drawing the tessellations (and tessellation by trapezoid figures was too difficult to draw for them), the other pairs used the paper models from the beginning of the interview; only the pair M+P1 used also the scissors for some additional problem emerging during the interview. All pairs found at least two tessellation arrangements for squares, rectangles and parallelograms (M+P1 was the most successful pair). In the solutions of pairs M+P1 and T+K there also occurred the using of the previous solutions (rectangle: “it is like the square, but there are two joined squares now”; rhomboid: “it is similar like by rectangles”; trapezoid: “when we put them like that, then a parallelogram is formed there” ), the other pairs tiled randomly (by my feeling). Handling the general convex and nonconvex quadrilaterals was the most difficult for children. Three pairs were not successful with one of them; after being told that the problem can be solved, they tried to find it and being again unsuccessful, they at least admitted “...maybe it could, but we would need more time to think about the arrangement...”. The pair G+P2 was successful with a general nonconvex quadrilateral, but not with a general convex one; they found the principle 'the shorter sides together, the longer sides together', but they were not be able to use it for the general convex type again. Seven children answered the question Which quadrilateral figure does seem to be the most interesting and why? that just a general nonconvex quadrilateral is interesting (or “strange”) because e.g. “it is absolutely something else like for example the square”. (Manipulating with the nonconvex quadrilaterals was very interesting process. It is not a “common” quadrilateral occurring
in mathematical lessons. Some people have to count the number of its sides to persuade themselves that this figure is really a quadrilateral.)

Three pairs used right the terminology of the quadrilaterals (their names were the continuous component of their slang language), two youngest pairs (K+T and M+P1) made some mistakes. But only these pairs intuitively found out why every quadrilateral tessellates the plane by itself: the underlined sentence from interview (M+P1) “So that if you have these angles, so you can put them together.” and (T+K) “If we put the shorter sides together and the longer ones, too... and we get the circle.”

The proofs were absent in these interviews, there was not the need of them because children thought they worked with all possible quadrilateral shapes (“and the other quadrilaterals look like as well as those ones”).

**Tessellations created by pupils**

This part will be the further subject of a research in which I shall try to answer the following questions:

What polygonal shapes and their arrangements do they use?

Why do they use these polygons?

What tool do they use to draw it?

Were they inspired by typical real life's tessellation (e.g. usual parquetings, honeycomb, ...)?

I worked with five pairs of children, but the drawings of the tessellations were turned in only from three pairs. Two girls (T+K) worked together, the other children prepared their drawings separately. This part of the homework was based on the working with the polygons different from the quadrilaterals. In spite of this fact P1 drew the tessellation by the nonconvex quadrilaterals. The simplest polygon, triangle, occurred only in one case (H). Children used to draw black pens and black pencils. K+T drew the tessellations on the square paper, M and P1 on the line paper and H on the white paper.

Hana's tessellations
CONCLUSIONS

The examined topic is very close to the real life (architecture, art) and its principles are based on mathematics. During the interviews children dealt with something absolutely common to them (square tiles, rectangular parquets) but they had not thought about it before. They are not used to use mathematics as a tool to solve the real life's problems yet.

Children were interested in the required problems, they answered the questions

*Did you like the solved problems? Are they different from those ones solved in Mathematics lessons? What is the difference?*

as follows:

*I liked the problems, mainly because statements have to be defended and to be proved. We don't do such problems, only sometimes we combine and glue the paper figures. Yes, they were interesting, perhaps we rather count in mathematics lessons. The problems were different from usual mathematical lessons. We count more than play with patterns on the lessons.*

Mathematics of bathroom's tilings or floor's parquetings is somewhat
unexpected even for the adults, too. There was held a didactic program for teachers of mathematics in February 2004 at our faculty called Two Days with Didactics of Mathematics. I contributed to it by a workshop entitled Parquetings, tilings, mosaics and geometry. Some participants of this workshop solved the same problems similar like the children - randomly. After solving the same problems, they were asked to answer the question “is it possible to use identical parquets of an arbitrary quadrilateral shape?” and they were obliged to use the words common for the children taught by them. I had already heard similar answers from the pupils in my previous experiences:

*It can, because we tested all “sorts” of quadrilaterals.*

*It can, because some pupils were successful in the class (I wasn't) and the teacher praised them.*

*Yes, it can, because I've been successful until now.*

It seems that even when of wide practical application the problem of tessellations is troublesome. In general, perhaps because of its complete absence in mathematical curriculum.

REFERENCES